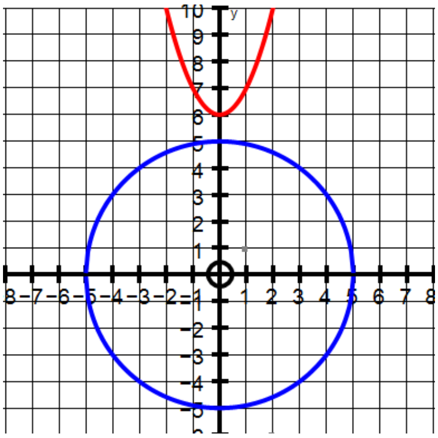
Intersection Points of Circles with Parabolas.

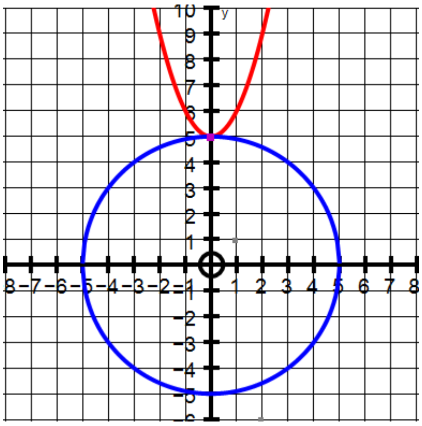
Consider various cases of the curves $y = x^2 + c$ and $x^2 + y^2 = 25$

1. When $c = 6$ we have $y = x^2 + 6$ and $x^2 + y^2 = 25$



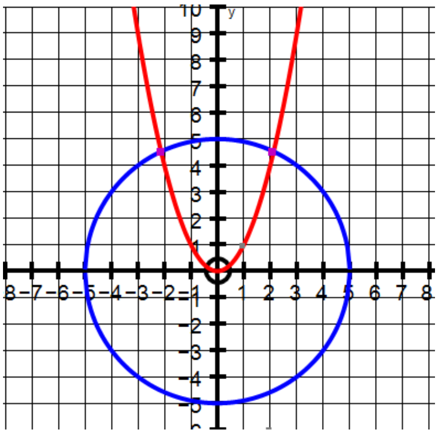
Here we see that there are apparently no intersection points.

2. When $c = 5$ we have $y = x^2 + 5$ and $x^2 + y^2 = 25$



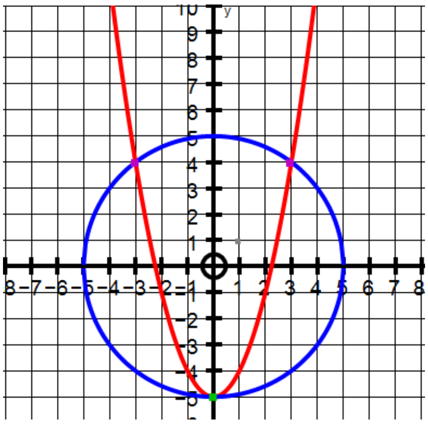
Here we see that there is apparently 1 intersection point at (0, 5)

3. When $c = 0$ we have $y = x^2 + 0$ and $x^2 + y^2 = 25$



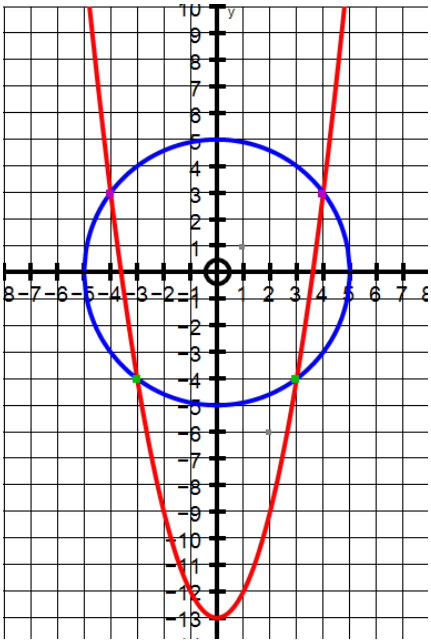
Here we see that there are apparently 2 intersection points at (2.1, 4.4) and at (–2.1, 4.4)

4. *When $c = -5$ we have $y = x^2 - 5$ and $x^2 + y^2 = 25$*



Here we see that there are apparently 3 intersection points at (3, 4) and (– 3, 4) and (0, – 5)

5. *When $c = -13$ we have $y = x^2 - 13$ and $x^2 + y^2 = 25$*



Here we see that there are apparently 4 intersection points at (4, 3) and (– 4, 3) and (3, – 4) and (– 3, – 4)

Solving these equations algebraically:

Subs $y = x^2 - 13$ into $x^2 + y^2 = 25$

$$x^2 + (x^2 - 13)^2 = 25$$
$$x^2 + x^4 - 25x^2 + 169 = 25$$
$$x^4 - 24x^2 - 144 = 0$$
$$(x^2 - 16)(x^2 - 9) = 0$$
$$(x + 4)(x - 4)(x + 3)(x - 3) = 0$$

So $x = \pm 4$ and ± 3
Intersection points are:
(–4, 3), (–3, –4), (3, –4), (4, 3)

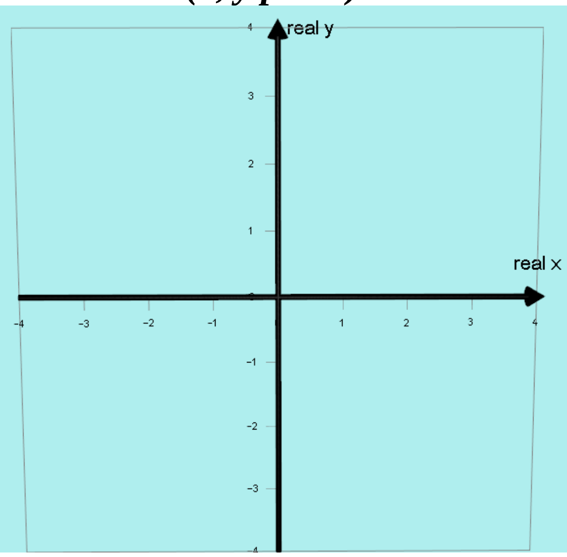
It occurred to me that if we repeat this algebra for the general case $y = x^2 + c$ we will always get a <u>quartic equation</u> to solve which will always have 4 solutions.
Subs $y = x^2 + c$ into $x^2 + y^2 = 25$

$$x^2 + (x^2 + c)^2 = 25$$
$$x^2 + x^4 + 2cx^2 + c^2 = 25$$
$$x^4 + (2c + 1)x^2 + (c^2 - 25) = 0$$

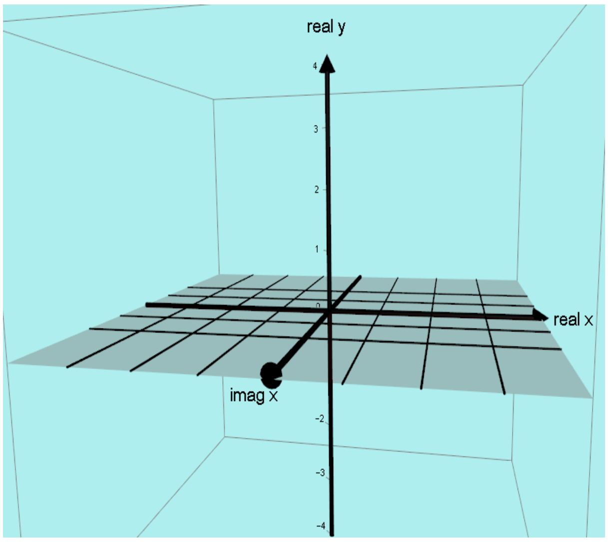
Logically, if we always should get 4 solutions for different values of c, there must always be 4 intersections which seems contrary to the examples above!

The question is "Where are the missing intersections?"
The clue of course is in the fact that some of the solutions to the quartic equations are complex numbers.
We cannot plot complex points on our ordinary 2Dimensional x, y axes.
We need to add an imaginary axis in order to produce a complex x plane with an ordinary y axis sticking up in the middle!

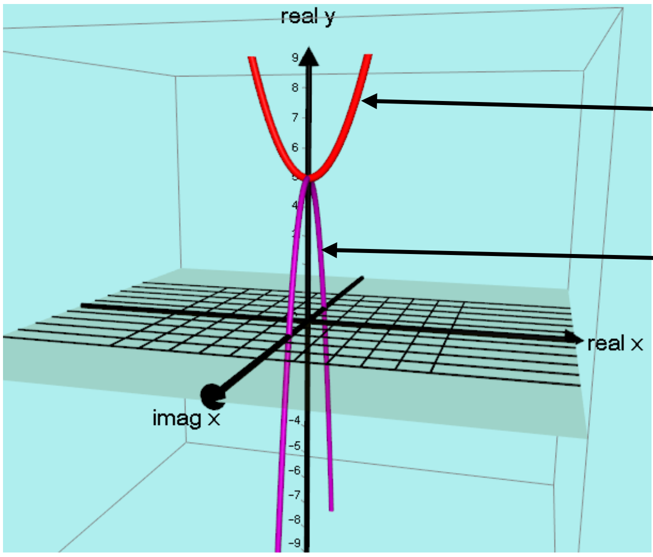
2D (x, y plane)



3D (complex x plane, real y axis)



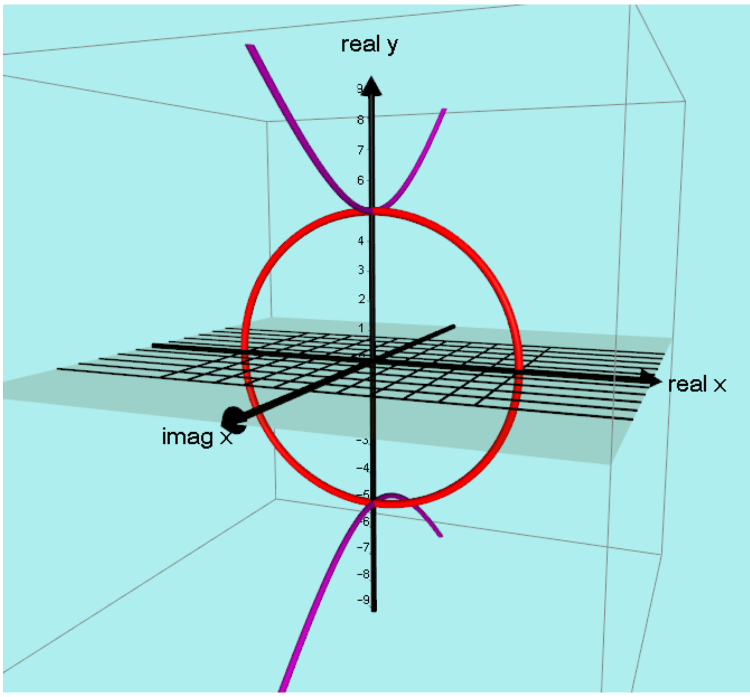
When we find complex x values which still produce real y values the Parabola has an extra part hanging from its minimum point.
(For a detailed explanation of this see www.phantomgraphs.weebly.com)



Basic parabola (red)

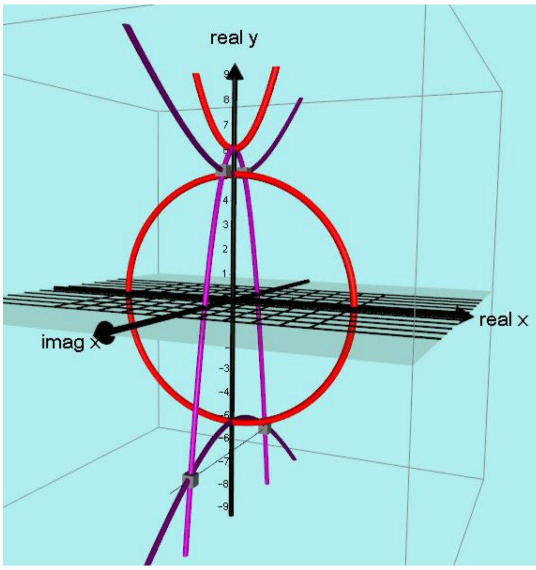
Phantom parabola (purple) is at right angles to the original.

Similarly, when we find complex x values which still produce real y values the Circle has 2 phantoms which are actually the two halves of a hyperbola!



Now I will add both phantom graphs and reconsider the examples at the start of this paper.

1. $y = x^2 + 6$ *and* $x^2 + y^2 = 25$



Subs $y = x^2 + 6$ *into* $x^2 + y^2 = 25$

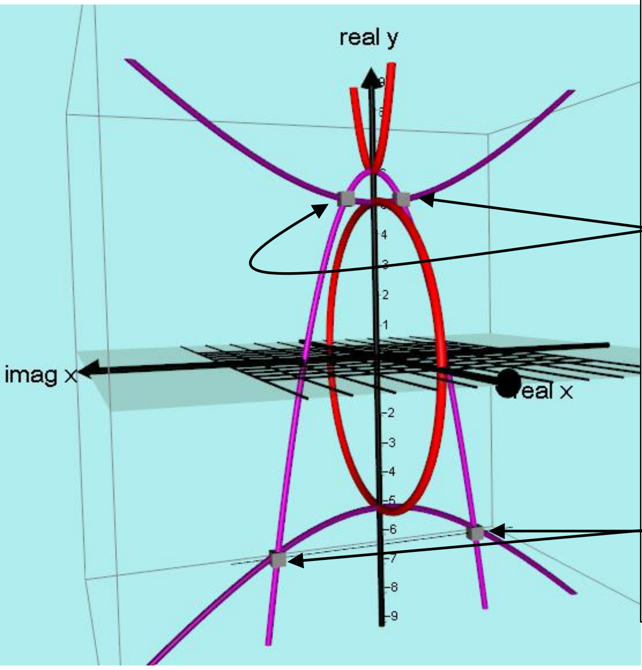
We get $x^2 + (x^2 + 6)^2 = 25$

$$x^2 + x^4 + 12x^2 + 36 = 25$$

$$x^4 + 13x^2 + 11 = 0$$

$x = 3.5i, -3.5i, 0.95i, -0.95i$

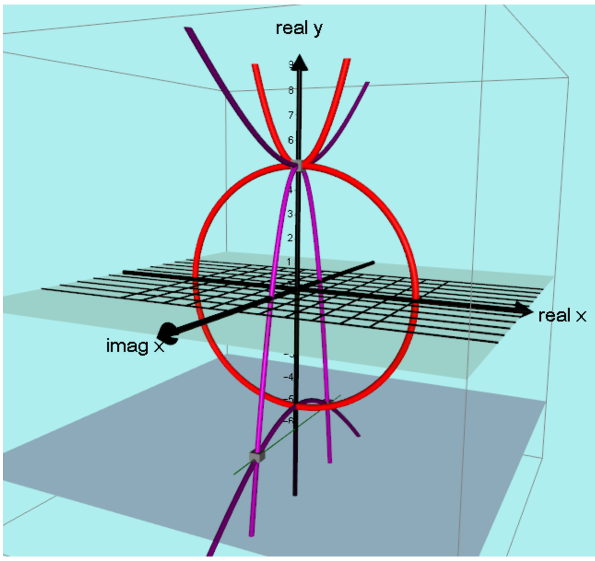
Rotated view:



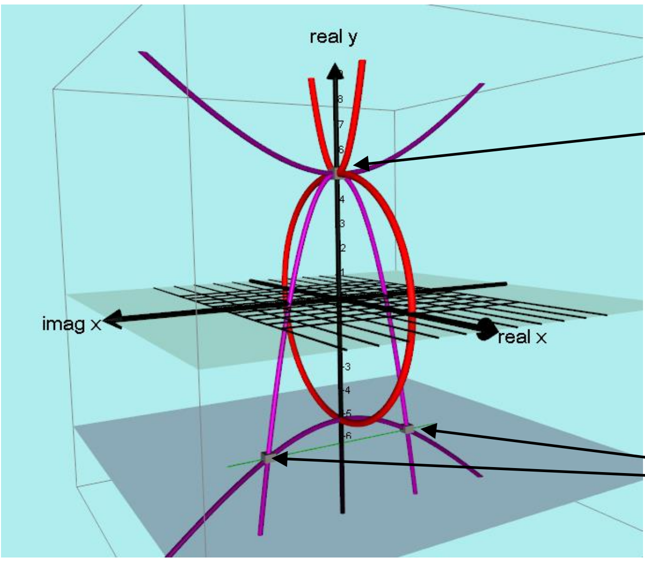
The parabola's phantom intersects the circle's upper phantom at these 2 points where $x = 0.95i$ *and* $-0.95i$

The parabola's phantom intersects the circle's lower phantom at these 2 points where $x = 3.5i$ *and* $-3.5i$,

2. $y = x^2 + 5$ and $x^2 + y^2 = 25$



Rotated view



Subs $y = x^2 + 5$ ***into*** $x^2 + y^2 = 25$

We get $x^2 + (x^2 + 5)^2 = 25$

$$x^2 + x^4 + 10x^2 + 25 = 25$$

$$x^4 + 11x^2 = 0$$

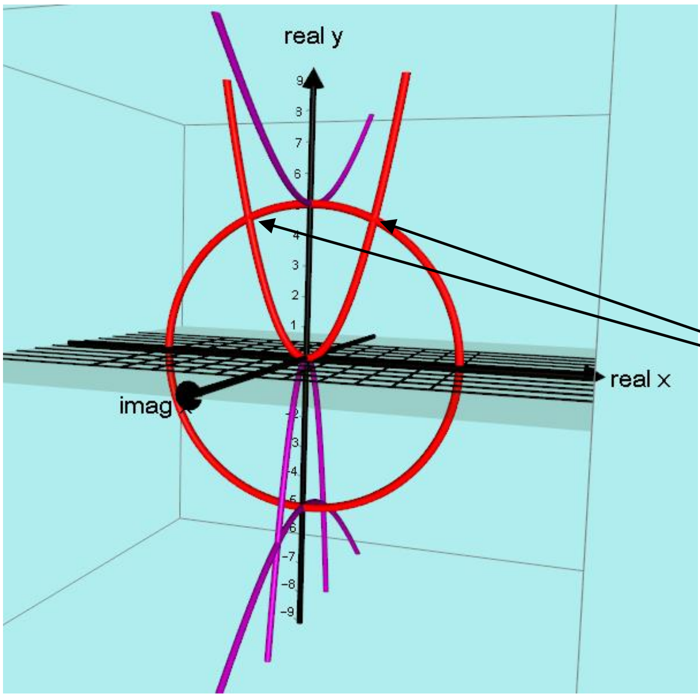
$$x^2(x^2 + 11) = 0$$

so $x = 0$ ***or*** $\pm\sqrt{11}$

Notice, where x = 0 the circle intersects with the parabola and the parabola's phantom also intersects with the circle's upper phantom. So this is a double intersection at (0, 5)

The parabola's phantom intersects the circle's lower phantom at these 2 points $(\sqrt{11}, -6)$ ***and*** $(-\sqrt{11}, -6)$

3. $y = x^2$ and $x^2 + y^2 = 25$



Subs $y = x^2$ into $x^2 + y^2 = 25$

We get $x^2 + (x^2)^2 = 25$

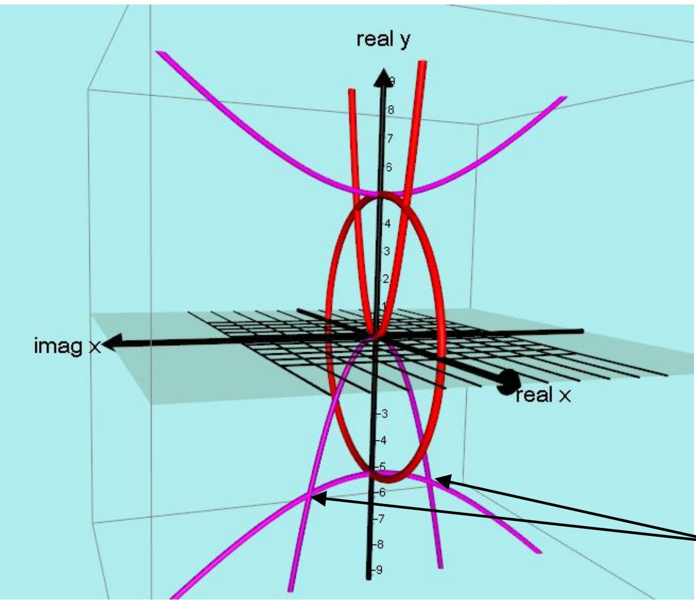
$$x^2 + x^4 = 25$$

$$x^4 + x^2 - 25 = 0$$

$$x = \pm 2.13, \pm 2.35i$$

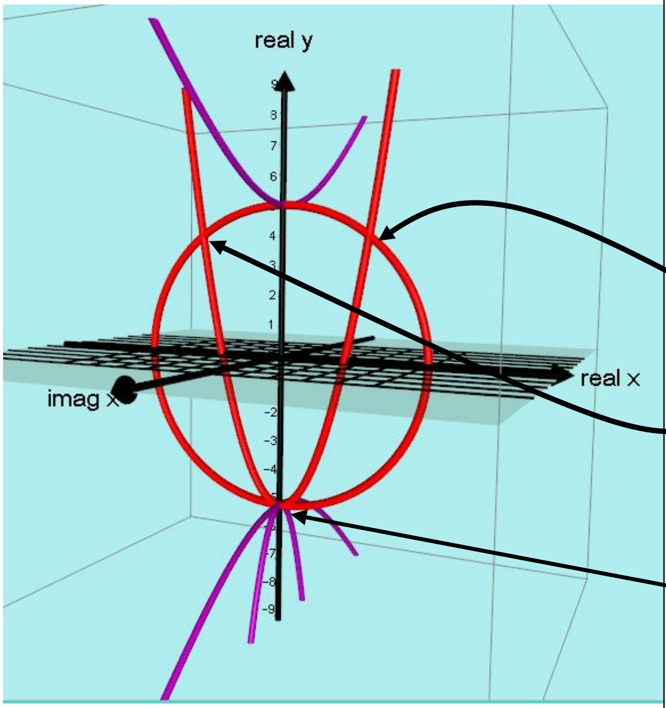
Notice there are 2 real intersections when $x = \pm 2.13$

Rotated view



Also notice the parabola's phantom intersects with the lower phantom of the circle at $x = \pm 2.35i$

4. ***y = x² – 5 and x² + y² = 25***



Subs $y = x^2 - 5$ ***into*** $x^2 + y^2 = 25$

We get $x^2 + (x^2 - 5)^2 = 25$

$$x^2 + x^4 - 10x^2 + 25 = 25$$

$$x^4 - 9x^2 = 0$$

$$x^2(x^2 - 9) = 0$$

$$x^2(x + 3)(x - 3) = 0$$

$$x = 0,\ 3 \text{ or } -3$$

Intersection points are:

$(-3, 4)$, $(3, 4)$ ***and*** $(0, -5)$

Notice the intersections here are all real on the x, y plane.

Again there is a “double” solution at (0, –5) because the curves intersect twice at this point. The original red curves intersect and the phantoms also intersect.

Also see the screencast video:

http://screencast.com/t/3bdbGbI1u